

# Non-Gaussian Path Integration in Self-Interacting Scalar Field Theories

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In self-interacting scalar field theories kinetic expansion is an alternative way of calculating the generating functional for Green's functions where the zeroth order non-Gaussian path integral becomes diagonal in  $x$ -space and reduces to the product of an ordinary integral at each point which can be evaluated exactly. We discuss how to deal with such functional integrals and propose a new perturbative expansion scheme which combines the elements of the kinetic expansion with that of usual perturbation theory. It is then shown that, when the cutoff dependent bare parameters in the potential are fixed to have a well defined non-Gaussian path integral without the kinetic term, the theory becomes trivial in the continuum limit.

## I. INTRODUCTION

In standard model, the scalar Higgs boson is the only fundamental particle that has not been discovered yet. Our understanding of the scalar sector is based on rather indirect observations. For instance, one loop radiative corrections in electroweak theory involve Higgs particle and the agreement between the theory and the experiment can be used to estimate the Higgs mass which is expected to be less than 225 GeV [1]. Also the scalar field is supposed to break the  $SU(2) \times U(1)$  gauge symmetry down to a  $U(1)$  subgroup by acquiring a non-zero vacuum expectation value around 247 GeV (determined from the experimental value of the W-boson mass and the Fermi coupling constant).

Apart from these and some other similar inputs, the scalar sector has not been tested directly. There is no experimental data which may be helpful in understanding the form and the strength of scalar self interactions. Specifically, the scalar potential remains to be determined. Assuming that the interactions are renormalizable and tractable in perturbation theory, the potential should at most be a quartic polynomial. Further requiring spontaneous symmetry breaking, the potential can uniquely be fixed to have the form  $\lambda(\phi^2 - \mu^2)^2$ .

However, perturbative quantization of scalar fields in standard model seems unnatural. Namely, there are quantum corrections to Higgs mass which are quadratic in cutoff  $\Lambda$ . Therefore, the mass parameter depends very sensitively on the short distance behavior of the theory and the value of the bare mass should be very fine tuned to maintain Higgs mass within a reasonable range. This is the so called naturalness or hierarchy problem.

Moreover, a perturbative renormalization of the self-interacting  $\lambda\phi^4$  theory following the ideas of Wilson indicates that the theory is free or trivial in the continuum limit [2]. Although a non-perturbative investigation of

the theory on the lattice agrees with naive one-loop perturbative result for renormalization group flow [3], it is also possible to argue that in the large  $N$  limit there is a non-trivial fixed point in the four dimensional scalar field theory [4].

Therefore, it is of great importance to develop new analytical methods alternative to usual perturbation theory which may help us to resolve these and similar issues in scalar field theories. In this paper, we consider one such framework, known as kinetic (or strong coupling) expansion, where the interaction potential, which can be a non-polynomial function, is treated exactly. The main strategy here is to deal with the kinetic term perturbatively and carry out Euclidean path integration in  $x$ -space (defined with a cutoff), where the potential is "diagonal" and the functional integral reduces to the product of an ordinary integral at each point (see e.g. [6]-[16]).

In this work we first discuss how to calculate a non-Gaussian functional integral which appears in the zeroth order contribution in the kinetic expansion. It turns out that the bare parameters in the potential should be taken cutoff dependent to get a finite result (see e.g. [7]). However, the higher order corrections in the expansion become ill defined due to the presence of uncontrollable infinities. One can try to perform an extrapolation to cure the problem as proposed in [10, 12], however this procedure likely fails in general as discussed in [11, 14]. The main obstacle in this program is that there is an expansion in derivatives about a zeroth order configuration which is calculated without paying attention to locality since the derivative (kinetic) term in the action is removed [15].

In section III, an alternative expansion scheme is proposed where one can overcome this obstacle. We find that by introducing a Lagrange multiplier it is possible to invert the kinetic term as in a free theory while the potential can also be integrated out non-perturbatively (but order by order in cutoff) yielding a complementary series. In section IV, we use this result to show that if the cutoff dependences of the bare parameters in the potential are chosen specifically to have a well defined non-Gaussian

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path integral without the kinetic term, then the theory becomes trivial in the continuum limit.

## II. PERTURBATIVE EXPANSION OF THE KINETIC TERM

In this section, we mainly review the kinetic expansion of the generating functional (see e.g. [10]). Consider a self-interacting, real scalar field theory defined in  $n$  dimensional space-time. The Euclidean path integral for the generating functional can be written as

$$Z[J] = N \int D\phi e^{\left[ \int \beta (\phi \square \phi) - V(\phi) + J\phi \right]}, \quad (1)$$

where  $\int$  means integration over  $d^n x$  (the  $x$  dependence of functions are suppressed)  $N$  is the (infinite) normalization,  $\beta$  is a constant and  $V(\phi)$  is the interaction potential including a possible mass term. We take  $n > 1$ , since we will freely play with the constant  $N$  in (1). This is not possible when  $n = 1$  since in quantum mechanics the integration measure is uniquely fixed by the normalization of the state vectors.

We evaluate (1) by expanding the exponential of the kinetic term so that  $\beta$  is the perturbation parameter. The zeroth order contribution is

$$Z_0[J] = N \int D\phi e^{\left[ \int -V(\phi) + J\phi \right]}. \quad (2)$$

We first approximate such functional integrals by dividing  $n$ -dimensional space into regions of volume  $\epsilon$ . One can think that the theory is defined with

$$\epsilon = L^n, \quad \Lambda = 1/L, \quad (3)$$

where  $L$  is the spatial lattice size and  $\Lambda$  is the cutoff in momentum space. In this approximation

$$\int f(x) d^n x \rightarrow \sum_i f_i \epsilon, \quad (4)$$

$$\frac{\delta}{\delta f(x)} \rightarrow \frac{\partial}{\partial f_i}, \quad (5)$$

and path integral measure is given by

$$D\phi \rightarrow \prod_i d\phi_i, \quad (6)$$

where the index  $i$  runs over the infinitesimal regions. The Dirac delta function  $\delta(x-y)$  and the derivative operator can be represented as  $\delta_{ij}/\epsilon$  and  $(\partial\phi)_i = (\phi_{i+1} - \phi_i)/\epsilon$ . We will keep  $\epsilon$  infinitesimal but finite and let  $\epsilon \rightarrow 0$  to reach the continuum limit.

Eq. (2) can now be rewritten as

$$\begin{aligned} Z_0[J] &= N \int \prod_i d\phi_i e^{\sum_i [-V(\phi_i) + J_i \phi_i] \epsilon} \\ &= N \prod_i z_i(J_i), \end{aligned} \quad (7)$$

where

$$z_i(J_i) = \int_{-\infty}^{\infty} d\phi_i e^{-V(\phi_i) \epsilon + J_i \phi_i \epsilon}. \quad (8)$$

Thus the problem reduces to the calculation of an ordinary integral. Although for simple polynomial potentials the exact result of (8) can be found,<sup>1</sup> for our purposes a series solution in  $J_i$  is enough. Of course the interaction potential should ensure the finiteness of each term and the convergence of the sum in the series. We take  $V$  to be an even function  $V(-\phi) = V(\phi)$  with  $V(\phi) \rightarrow +\phi^p$  for some  $p > 1$  as  $\phi \rightarrow \pm\infty$  to satisfy this technical condition. Let us scale  $\phi_i \rightarrow \phi_i/\sqrt{\epsilon}$  in (8) to organize the expansion. It is important to note that any overall multiplicative constant (like  $1/\sqrt{\epsilon}$  which appears after the scaling) can be absorbed in the definition of the constant  $N$  in (7) and therefore *ignored*. Introducing

$$\widehat{V}(\phi_i) = V(\phi_i/\sqrt{\epsilon}) \epsilon, \quad (9)$$

and

$$a_m = \frac{1}{m!} \int_{-\infty}^{\infty} e^{-\widehat{V}(u)} u^m du, \quad (10)$$

(8) can be evaluated as

$$z_i(J_i) = 1 + \sum_{m=1}^{\infty} \frac{a_{2m}}{a_0} J_i^{2m} \epsilon^m, \quad (11)$$

where  $a_0$  is also factored out to ensure the normalization  $z_i(0) = 1$ . Note that only even powers of  $J_i$  survive since we take  $V$  to be an even function.

Using (11) in (7) one obtains

$$\begin{aligned} Z_0[J] &= \prod_i \left[ 1 + \frac{a_2}{a_0} J_i^2 \epsilon + \frac{a_4}{a_0} J_i^4 \epsilon^2 + \dots \right], \\ &= \exp \left[ \sum_i \frac{a_2}{a_0} J_i^2 \epsilon \right] + \mathcal{O}(\epsilon), \end{aligned} \quad (12)$$

and in the continuum limit we get

$$Z_0[J] = \exp \left[ \int \frac{a_2}{a_0} J(x)^2 d^n x \right]. \quad (13)$$

Although (13) resembles a functional obtained after a Gaussian path integration, there is a crucial difference;

<sup>1</sup> For  $V = \lambda\phi^4$  the result of (8) is

$$\frac{\pi}{8\sqrt{2}(\epsilon\lambda)^{3/4}} \times \left( \frac{8\sqrt{\epsilon\lambda}}{\Gamma[\frac{3}{4}]} {}_0F_2 \left[ \frac{1}{2}, \frac{3}{4}; \frac{\epsilon^3 J_i^4}{256\lambda} \right] + \frac{(\epsilon J_i)^2}{\Gamma[\frac{5}{4}]} {}_0F_2 \left[ \frac{5}{4}, \frac{3}{2}; \frac{\epsilon^3 J_i^4}{256\lambda} \right] \right),$$

where  ${}_pF_q[\mathbf{a}; \mathbf{b}; x]$  is the generalized hypergeometric function which is regular at  $x = 0$  and normalized so that  ${}_pF_q[\mathbf{a}; \mathbf{b}; 0] = 1$ . A standard integral table can be found in [5]

there are actually  $\mathcal{O}(\epsilon)$  corrections to (13) which can be seen in (12). These can be ignored at this point, however they are not negligible when  $\beta$  corrections are calculated (see below).

The constants  $a_0$  and  $a_2$  might depend on the cutoff so there can still be infinities hiding in  $Z_0[J]$ . To get a finite result in the continuum,  $\hat{V}$  defined in (9) should not depend on  $\epsilon$  as  $\epsilon \rightarrow 0$ . This can be ensured by taking the bare parameters in the potential cutoff dependent.<sup>2</sup> For instance, if

$$V = m^2 \phi^2 + \lambda \phi^p, \quad (14)$$

then for  $\hat{V}$  to become cutoff independent  $\lambda$  should be fixed as

$$\lambda = \left(\frac{\epsilon}{\epsilon_\mu}\right)^{(p-2)/2} \lambda_\mu = \left(\frac{\mu}{\Lambda}\right)^{n(p-2)/2} \lambda_\mu, \quad (15)$$

where  $\epsilon_\mu$  corresponds to a fixed scale  $\mu$ , i.e.  $\epsilon_\mu = \mu^{-n}$ , and  $\lambda_\mu$  is the “renormalized” coupling constant. Using (15) the cutoff  $\Lambda$  and bare coupling  $\lambda$  dependences of  $a_0$  and  $a_2$  can be traded with  $\mu$  and  $\lambda_\mu$ , respectively, which remain finite in the continuum limit. Note that for the bare mass  $m$  there is no need to impose any cutoff dependence. Eq. (15) can be viewed as the analog of the renormalization group flow in this framework. For  $p > 2$  the bare coupling vanishes and for  $p < 2$  it diverges in the continuum limit  $\epsilon \rightarrow 0$  or  $\Lambda \rightarrow \infty$ .

To illustrate the above formulas with a specific example, let us take  $V = m^2 \phi^2 + \lambda \phi^4$ . Then the constants  $a_0$  and  $a_2$  can be found using (10) which gives

$$a_0 = \sqrt{\frac{x}{m^2}} e^{x/2} K_{1/4}(x/2), \quad (16)$$

$$a_2 = \frac{1}{4} \left(\frac{\epsilon_\mu}{\lambda_\mu}\right)^{3/4} \Gamma[3/4] {}_1F_1[3/4, 1/2; x] - \frac{m^2}{4} \left(\frac{\epsilon_\mu}{\lambda_\mu}\right)^{5/4} \Gamma[5/4] {}_1F_1[5/4, 3/2; x], \quad (17)$$

where  $x = m^4 \epsilon_\mu / (4 \lambda_\mu)$ ,  $K$  is the Bessel and  ${}_pF_q[\mathbf{a}; \mathbf{b}; x]$  is the generalized hypergeometric function (see also footnote 1). From these, it is clear that  $\lambda_\mu$  dependence of (13) is not analytic and cannot be obtained in usual perturbation theory.

The full generating functional (1) can formally be expressed as

$$Z[J] = \exp \left[ \beta \int G(x, y) \frac{\delta^2}{\delta J(x) \delta J(y)} d^n x d^n y \right] Z_0[J] \quad (18)$$

where  $G(x, y) = \square_x \delta(x - y)$  and  $Z_0[J]$  is given in (7). Eq. (18) is ready for a perturbative expansion in  $\beta$  where the zeroth order contribution is given by (13). It is important that one uses (7) (together with (11)), but *not* simply (13), for  $Z_0[J]$  in (18) to calculate higher order corrections. This is because, from (5), the functional derivatives have a (singular) cutoff dependence and when acting on  $\mathcal{O}(\epsilon)$  corrections in (12) this would yield non-vanishing contributions in the continuum limit (this is actually the case as we will see).

Expanding (18) to first order in  $\beta$  and using (7) one can find

$$Z[J] = Z_0[J] + \beta Z_0[J] \sum_{ij} G_{ij} \frac{\partial \ln z_i}{\partial J_i} \frac{\partial \ln z_j}{\partial J_j} + \beta Z_0[J] \sum_i G_{ii} \frac{\partial^2 \ln z_i}{\partial J_i^2} + \mathcal{O}(\beta^2) \quad (19)$$

where  $G_{ij}$  denote the entries of  $G(x, y)$  on the lattice. Note that  $\epsilon^2$  factor coming from the integration measure  $d^n x d^n y$  in (18) is canceled by the cutoff dependence of the functional derivatives (5).

Let us now take the continuum limit. Using  $z_i$  given in (11) we see that the leading order  $\epsilon$  terms in (19) are  $\epsilon^2$  and  $\epsilon$  for the first and the second lines, respectively, which are exactly the required powers to convert sums into integrals. Unfortunately, however, the diagonal entries of  $G_{ij}$  diverge for the kinetic term since in the continuum we have  $G(x, y) = \square_x \delta(x - y)$ , so there is a possible divergence problem in the second line in (19).

For the moment, assume that  $G(x, y)$  is a smooth and finite function. Then in  $\epsilon \rightarrow 0$  limit (19) becomes

$$Z[J] = (1 + \beta C) Z_0[J] + 4\beta \left[\frac{a_2}{a_0}\right]^2 Z_0[J] \int G(x, y) J(x) J(y) d^n x d^n y, \quad (20)$$

where  $C = 2(a_2/a_0) \int G(x, x) d^n x$  is a constant. The properly normalized generating functional up to  $\mathcal{O}(\beta^2)$  is

$$\frac{Z[J]}{Z[0]} = Z_0[J] + 4\beta \left[\frac{a_2}{a_0}\right]^2 Z_0[J] \int G(x, y) J(x) J(y) d^n x d^n y, \quad (21)$$

so the dependence on a (possibly infinite) contribution  $C$  drops out (not that we set  $Z_0[0] = 1$ ). Of course one can now use (13) in (21) since there is no functional derivative acting on  $Z_0[J]$ . The constant  $C$  is related to “vacuum to vacuum” amplitude and it may also be eliminated by redefining the normalization  $N$  in (7). Thus, the first order result (21) is finite in the continuum for smooth  $G(x, y)$ .

To see what happens for a singular function let us take  $G(x, y) = -m^2 \delta(x - y)$ . Since the mass term can actually be treated exactly, the perturbative first order result

<sup>2</sup> Recall that in usual perturbation theory cutoff dependences of the bare parameters in the Lagrangian are fixed by requiring finiteness of physical quantities. Here we employ the same philosophy.

may be derived from the exact formula by expanding the coefficient of  $J(x)^2$  in (13) around  $m = 0$  for the potential  $V = V_0 + m^2 \phi^2$ . Using, on the other hand, the lattice approximation  $G_{ij} = -m^2 \delta_{ij}/\epsilon$  in (19) one can directly get

$$\frac{Z[J]}{Z[0]} = Z_0[J] + 2\beta m^2 \left[ \frac{a_2^2}{a_0^2} - \frac{6a_4}{a_0} \right] Z_0[J] \int J(x)^2 d^n x. \quad (22)$$

As in (20), there is a  $CZ_0[J]$  term in  $Z[J]$  which is eliminated after dividing by  $Z[0]$  where the constant  $C$  diverges like  $C \sim \int d^n x$ .

For the kinetic term the situation is more complicated. Using the approximations for the delta function and the derivative operator given below (6) one finds  $G_{ij} = (\delta_{i+2,j} - 2\delta_{i+1,j} + \delta_{ij})/\epsilon^3$ . Therefore  $G_{ii} = 1/\epsilon^3$  (no sum in  $i$  is implied). However, it is also possible to define an asymmetric derivative operator such as  $\hat{\partial}\phi_i = (\phi_{i+1} - \phi_{i-2})/(3\epsilon)$  which gives  $G_{ij} = (\delta_{i+2,j-2} - 2\delta_{i-1,j} + \delta_{i-4,j})/(9\epsilon^3)$  and thus  $G_{ii} = 0$ . Note that  $(\hat{\partial}\phi)_i = (\partial\phi_i + \partial\phi_{i-1} + \partial\phi_{i-2})/3$  so  $\hat{\partial}$  involves an extra averaging over neighboring lattice points in terms of  $\partial$ . In the continuum limit both  $\partial$  and  $\hat{\partial}$  asymptote to the same differential operator. But, one should actually specify the function space in the continuum more precisely. Namely,  $\hat{\partial}$  can only be used as the derivative operator if the test functions are not changing appreciably along three neighboring lattice points.

To resolve the issue we use the Fourier transform with a momentum cutoff  $\Lambda$  to specify smoothness of the field variables in the continuum. From

$$G(x, y) = - \int e^{ip(x-y)} p^2 d^n p, \quad (23)$$

one gets  $G(x, x) \sim - \int^\Lambda p^{n+1} dp \sim -\Lambda^{n+2}$  and thus we take  $G_{ii} = c\epsilon^{-(n+2)/n}$  in (19) where  $c$  is a (finite) numerical constant. In the continuum limit we then have

$$\frac{Z[J]}{Z[0]} = Z_0[J] + 4\beta \left[ \frac{a_2}{a_0} \right]^2 Z_0[J] \int J(x) \square_x J(x) d^n x - 6\beta c \epsilon^{-2/n} \left[ \frac{2a_4}{a_0} - \frac{a_2^2}{a_0^2} \right] Z_0[J] \int J(x)^2 d^n x. \quad (24)$$

Unfortunately, the second line of (24) diverges like  $\epsilon^{-2/n} \sim \Lambda^2$ . However, from (22), this divergence can be canceled out by a mass counterterm  $(\delta m)^2 \sim \beta \Lambda^2$ . So we conclude that, to first order in  $\beta$ , only a mass renormalization is required after which all Green's functions become finite.

As for the higher order  $\beta$  corrections, it is not difficult to obtain the analog of (19) on the lattice. For smooth and finite  $G(x, y)$ ,  $\mathcal{O}(\beta^2)$  contribution turn out to be finite in the continuum. However, analyzing the result for the kinetic term to this order we find that some Green's functions diverge (e.g. quadratically in  $\Lambda$ ) and it seems

these cannot simply be removed by local counterterms as in (24). We also expect more and more infinities to show up in higher orders coming from the multiplications of the singular distribution  $\square_x \delta(x - y)$ . So it seems that new regularization or renormalization techniques should be developed for higher orders.<sup>3</sup> As pointed out in the introduction the main problem here is that in calculating  $Z_0[J]$  locality is neglected since the kinetic term in the action is removed. On the contrary, the higher order corrections are sensitive to local variations since more and more powers of the Laplacian appear in the expansion.

### III. AN ALTERNATIVE EXPANSION

Our main strategy in this section is to use a Lagrange multiplier in the path integral to separate the kinetic and the potential terms so that (1) can be rewritten as

$$Z[J] = \int D\phi D\Phi D\rho e^{\left[ \int \beta(\phi \square \phi) - V(\Phi) + J\Phi + i\rho(\phi - \Phi) \right]} \quad (25)$$

where the normalization  $N$  is suppressed. Evidently integrating over  $\rho$  gives a delta functional and a further trivial integration over  $\Phi$  gives (1). However one can now carry out the Gaussian  $\phi$  integration to get

$$Z[J] = \int D\Phi D\rho e^{\left[ \int \frac{1}{4\beta}(\rho \square^{-1} \rho) - V(\Phi) + J\Phi + i\rho(K - \Phi) \right]} \Bigg|_{K=0}$$

where we have introduced an auxiliary field  $K$ . Since differentiating with respect to  $K(x)$  gives  $i\rho(x)$ , the last equation can be expressed as

$$Z[J] = \exp \left[ -\frac{1}{4\beta} \int \frac{\delta}{\delta K(x)} \square_x^{-1} \frac{\delta}{\delta K(x)} d^n x \right] \times \int D\Phi D\rho e^{\left[ \int -V(\Phi) + J\Phi + i\rho(K - \Phi) \right]} \Bigg|_{K=0}.$$

At this point,  $\rho$  and  $\Phi$  integrations become trivial where the first one gives a delta functional and the second one

<sup>3</sup> One possibility is to set up a renormalization group flow starting from IR. For instance, from (23) one may decompose

$$G(x, y) = G(x, y, \mu) + \hat{G}(x, y),$$

where

$$G(x, y, \mu) = - \int_{-\mu}^{\mu} e^{ip(x-y)} p^2 d^n p,$$

$\mu$  is a fixed IR length scale and  $\hat{G}$  contains the integral in (23) starting from  $\mu$  to  $\infty$ . Now, in (18) one can first calculate the contribution of  $G(x, y, \mu)$ . Since  $G(x, y, \mu)$  is non-singular the exact result of the computation can in principle be performed to any given order without encountering any infinities or ambiguities. This gives a new functional  $Z_\mu[J]$  and one can now integrate the modes from  $\mu$  to  $2\mu$  to obtain  $Z_{2\mu}[J]$  etc.

sets  $\Phi = K$ . After these integrations one finds

$$Z[J] = e^{\left[-\frac{1}{4\beta} \int \frac{\delta}{\delta K} \square^{-1} \frac{\delta}{\delta K}\right]} e^{\left[\int -V(K) + JK\right]} \Big|_{K=0}, \quad (26)$$

which can be evaluated order by order in  $1/\beta$  by expanding the first exponential. Comparing to (18) where  $\beta$  is the expansion parameter, we see that (26) gives a complementary series, reminiscent of a strong-weak coupling duality.

It is not difficult to show that for polynomial potentials (26) is equivalent to usual perturbation theory. Noting that the functional derivative of  $\exp(\int JK)$  with respect to  $J(x)$  ( $K(x)$ ) gives  $K(x)$  ( $J(x)$ ), (26) can be rewritten as

$$\begin{aligned} Z[J] &= e^{-\frac{1}{4\beta} \int \frac{\delta}{\delta K} \square^{-1} \frac{\delta}{\delta K}} e^{\int -V(\frac{\delta}{\delta J})} e^{\int JK} \Big|_{K=0}, \\ &= e^{\int -V(\frac{\delta}{\delta J})} e^{-\frac{1}{4\beta} \int \frac{\delta}{\delta K} \square^{-1} \frac{\delta}{\delta K}} e^{\int JK} \Big|_{K=0}, \\ &= e^{\int -V(\frac{\delta}{\delta J})} e^{-\frac{1}{4\beta} \int J \square^{-1} J} e^{\int JK} \Big|_{K=0}, \\ &= e^{\int -V(\frac{\delta}{\delta J})} e^{-\frac{1}{4\beta} \int J \square^{-1} J}, \end{aligned} \quad (27)$$

where in the second line we have used the fact that  $J$  and  $K$  derivatives commute and in the last line  $\exp(\int JK)$  is dropped since any  $J$  derivative acting on it vanishes after setting  $K = 0$ . This proves the equivalence since (27) is nothing but the usual perturbative expansion formula. From the commutator

$$\left[ \int \frac{\delta}{\delta K} \square^{-1} \frac{\delta}{\delta K}, K(x) \right] = 2 \square^{-1} \frac{\delta}{\delta K(x)}, \quad (28)$$

one can also directly verify that (26) obeys Schwinger-Dyson equation

$$2\beta \square_x \frac{\delta Z[J]}{\delta J(x)} - V' \left( \frac{\delta}{\delta J(x)} \right) Z[J] + J(x) Z[J] = 0, \quad (29)$$

where  $V' = dV/d\phi$ . Note that in (26), for a fixed order in  $1/\beta$  (and for polynomial potentials), only a finite number of terms survive if one expands  $\exp(-\int V(K))$  since one sets  $K = 0$  at the end.

Eq. (26) is preferable over (27) for organizing the perturbative expansion in powers of  $1/\beta$ . Another advantage is that non-polynomial potentials can also be treated naturally. Assuming that  $V(K)$  is an infinitely differentiable smooth function, (26) shows that there is an interaction vertex for the  $p$ 'th derivative of  $V(K)$  if  $d^p V/dK^p \neq 0$  at  $K = 0$ . One may reach the same conclusion in perturbation theory by expanding  $V(K)$  around  $K = 0$ .

Having rederived the usual perturbation theory in a different way, we now proceed by obtaining another expression for the generating functional. Let us start with (25) in which the external current  $J$  is coupled to  $\phi$  rather than  $\Phi$ . Integrating over  $\Phi$  one gets

$$Z[J] = \int D\phi D\rho Z_0[i\rho] e^{\left[\int \beta (\phi \square \phi) + J\phi + i\rho(K-\phi)\right]} \Big|_{K=0},$$

where the functional  $Z_0$  is defined in (2) and we again introduced an auxiliary field  $K$ . Using the same trick we employed below (25), we find

$$Z[J] = Z_0 \left[ \frac{\delta}{\delta K} \right] e^{\left[\int \beta (K \square K) + JK\right]} \Big|_{K=0} \quad (30)$$

Adding and subtracting  $\int J \square^{-1} J/(4\beta)$  term in the exponential a complete square can be obtained. Shifting  $K$  as  $K(x) \rightarrow K(x) - \square_x^{-1} J(x)/(2\beta)$  one finally reaches

$$Z[J] = e^{\left[-\frac{1}{4\beta} \int J \square^{-1} J\right]} Z_0 \left[ \frac{\delta}{\delta K} \right] e^{\left[\beta \int K \square K\right]} \Big|_{2\beta \square K=J} \quad (31)$$

or more conveniently

$$Z[J] = e^{\left[-\frac{1}{2} \int J \square^{-1} J\right]} Z_0 \left[ \square \frac{\delta}{\delta J} \right] e^{\left[\frac{1}{2} \int J \square^{-1} J\right]}, \quad (32)$$

where we set  $\beta = 1/2$  since it can no longer be used as an expansion parameter (note  $1/\beta$  factors in (31)). Unlike the expression given in (18), we now manage to invert the kinetic term as in a Gaussian integral. Using the identity (34) below it is easy to verify (32) in the free theory when the potential contains only a mass term. Note that apart from the first exponential (and a sign in the third one)  $\log Z_0[\square\phi]$  can be viewed as an effective ‘‘potential’’ added to a free massless scalar field.

In the continuum limit, the functional  $Z_0$  is given in (13). However,  $\mathcal{O}(\epsilon)$  corrections are not negligible as before. Therefore, a new expansion scheme can be obtained by evaluating  $Z_0$  order by order in  $\epsilon$  and using the result in (31) or in (32).

Let us proceed by a direct computation of the first two terms in this expansion (a systematic treatment will be presented in the next section). From (12) we obtain

$$\begin{aligned} Z_0[f] &= \exp \left[ \int \frac{a_2}{a_0} f(x)^2 d^n x \right] \times \\ &\left\{ 1 + \kappa \epsilon \left( \frac{a_4}{a_0} - \frac{a_2^2}{2a_0^2} \right) \int f(x)^4 d^n x + \mathcal{O}(\kappa^2) \right\}, \end{aligned} \quad (33)$$

where  $\kappa$  is a formal expansion parameter which counts the powers of  $\epsilon$  (more comments on this below). To calculate the contribution of the exponential we note the identity

$$e^{\left[-\int \frac{A}{2} \frac{\delta^2}{\delta J^2}\right]} e^{\left[\int \frac{J^2}{2B}\right]} = e^{\left[\int \frac{J^2}{2(A+B)}\right]}, \quad (34)$$

which can be verified by setting up a simple Gaussian path integral. In (34), an overall irrelevant (infinite) multiplicative constant is omitted on the right hand side. Using (33) and (34) in (32) the zeroth order term can be found as

$$Z[J] = e^{\left[\frac{1}{2} \int J P_1 J\right]} + \mathcal{O}(\kappa), \quad (35)$$

where

$$P_1(x, y) = \left[ \frac{a_0}{2a_2} - \square \right]_x^{-1} \delta(x - y). \quad (36)$$

The same equations show that to calculate the next order contribution one confronts the term

$$\left\{ \int \left( \square_x \frac{\delta}{\delta J(x)} \right)^4 d^n x \right\} \exp \left[ \frac{1}{2} \int J \square^{-1} P_2 J \right], \quad (37)$$

where

$$P_2 = \frac{a_0}{2a_2} P_1. \quad (38)$$

The result of (37) contains the  $x$ -integral of

$$\begin{aligned} & 3 (\square P_2)_{xx}^2 \\ & + 6 (\square P_2)_{xx} \int P_2(x, y_1) P_2(x, y_2) J(y_1) J(y_2) dy_1 dy_2 \\ & + \int P_2(x, y_1) P_2(x, y_2) P_2(x, y_3) P_2(x, y_4) \times \\ & \quad J(y_1) J(y_2) J(y_3) J(y_4) dy_1 dy_2 dy_3 dy_4. \end{aligned} \quad (39)$$

The last line can be dropped in the continuum since, from (33), the whole expression is multiplied by  $\epsilon$ . For the first two lines, one needs the numerical value of  $(\square P_2)_{xx}$  which can be found from the Fourier transform

$$\square P_2(x, y) = - \int \frac{p^2}{1 + 2p^2 a_2/a_0} e^{ip(x-y)} d^n p. \quad (40)$$

When  $x = y$  the integral diverges like  $\Lambda^n$  and thus  $(\square P_2)_{xx} = c/\epsilon$  where  $c$  is a finite constant. The first line in (39) yields an infinity, however this can be pushed to order  $\kappa^2$  after dividing  $Z[J]$  by  $Z[0]$ . On the other hand  $1/\epsilon$  factor coming from  $(\square P_2)_{xx}$  in the second line is canceled by the  $\epsilon$  term in (33), thus there is no problem in taking  $\epsilon \rightarrow 0$  limit. Combining all these, the continuum result becomes

$$\begin{aligned} \frac{Z[J]}{Z[0]} = e^{\left[ \frac{1}{2} \int J P_1 J \right]} & \left\{ 1 + 6 c \kappa \left( \frac{a_4}{a_0} - \frac{a_2^2}{2a_0^2} \right) \times \right. \\ & \left. \int P_2(x, y_1) P_2(x, y_2) J(y_1) J(y_2) dx dy_1 dy_2 + \mathcal{O}(\kappa^2) \right\} \end{aligned} \quad (41)$$

which does not contain any infinities.

Higher order  $\kappa$  corrections can be calculated in a similar fashion where one now encounters more coincident functional derivatives acting in (32) and there are both connected and disconnected contributions. Of course, more coincident derivatives give a much more singular behavior. However, these are multiplied by higher powers of  $\epsilon$  and there is a chance that one gets a finite result in the continuum as for the first order calculation above. This is indeed the case as will be discussed in the next section.

The constant  $\kappa$  is just a formal expansion parameter and one should actually set  $\kappa = 1$  at the end. Therefore, the higher order contributions might in principle be greater than the lower order ones. However, by analyzing the expansion of  $Z_0$  we observe that higher order terms

contain  $a_k$  coefficients with larger  $k$ . Moreover, at least for polynomial potentials, a scaling argument shows that the coupling constant dependence of  $a_k$  has a hierarchy fixed by  $k$  i.e. for larger  $k$  one gets more powers of the inverse coupling constant (see e.g. the expressions given in (16) and (17)). Therefore, the expansion in  $\kappa$  can be reorganized as an expansion in the inverse coupling constant.

#### IV. TRIVIALITY

In section II, we found that to have a well defined non-Gaussian path integral in the kinetic expansion the bare coupling constant  $\lambda$  should obey (15). Therefore,  $\lambda$  vanishes for  $p > 2$  in the continuum limit as  $\Lambda \rightarrow \infty$ , and one may wonder whether the theory becomes free in this limit. Although the interactions vanish classically, this is not an obvious question in quantum theory. Indeed, there is no sign of triviality in the kinetic expansion studied in section II.

Let us try to analyze the situation in usual perturbation theory with the interaction potential (14) (so  $p$  is assumed to be an integer). Without loss of generality we take  $p > 2$  and focus on the connected graphs. It is well known that infinities generically arise in evaluating Feynman diagrams, however in our case these can be suppressed by the bare coupling constant since it vanishes like (15) as one removes the cutoff  $\Lambda$ . Consider a connected graph with  $I$  internal lines,  $E$  external lines,  $N$  vertices and  $L$  loops. We have the following topological identities:

$$2I + E = Np, \quad L = I - N + 1. \quad (42)$$

The superficial degree of divergence  $D$  of this diagram is given by (for each loop the momentum space integration measure contributes  $n$  units, and each internal line propagator adds  $-2$ )

$$D = nL - 2I. \quad (43)$$

Using (42) to eliminate  $L$  and  $I$  one gets

$$D = -\frac{nE}{2} - 2I + n + \frac{Nn(p-2)}{2}. \quad (44)$$

The order of the diagram is  $N$  and the degree of suppression  $S$  coming from the cutoff dependence of  $\lambda$  in (15) can be found as

$$S = \frac{Nn(p-2)}{2}. \quad (45)$$

Comparing the two numbers we find that  $S > D$ . Therefore, as  $\Lambda \rightarrow \infty$ ,  $\lambda^N$  term becomes more singular than the outcome of the momentum space integral and the final result for the diagram should tend to 0. According to this naive power counting, the only non-vanishing graph is the tree level two point function, i.e. the quantum theory is actually free. It is important to emphasize

that this is not a formal proof of triviality since the calculated degree of divergence  $D$  arises only from regions of momentum space in which all internal  $n$ -momenta go to infinity together. It is known that additional divergences can also show up when the momenta belonging to some subgraph tend to infinity. Therefore,  $D$  does not necessarily give the actual degree of divergence of a diagram. However, the above power counting is suggestive and indicates triviality.

Let us now try to use the alternative expression (31) in evaluating the generating functional. From (12) one finds

$$Z_0[f] = \exp \left[ \int \frac{a_2}{a_0} f(x)^2 d^n x \right] \times Z_{int}[f], \quad (46)$$

where  $Z_{int}[f]$  has an expansion of the form

$$Z_{int}[f] = \exp \left[ \sum_{k=2}^{\infty} \epsilon^{k-1} A_k \int f(x)^{2k} d^n x \right] \quad (47)$$

and  $A_k$  are finite coefficients which can be determined in terms of  $a_k$ , e.g.  $A_2 = a_4/a_0 - a_2^2/(2a_0^2)$ . Using (46) and (34) in (31), we get

$$Z[J] = \exp \left[ -\frac{1}{2} \int J \square^{-1} J \right] \times Z_{int} \left[ \frac{\delta}{\delta K} \right] e^{\left[ \frac{1}{2} \int K \Delta K \right]} \Big|_{\square K = J} \quad (48)$$

where

$$\Delta(x, y) = \square P_2(x, y). \quad (49)$$

From (47) the second line of (48) is equivalent to the generating functional of a scalar field  $\varphi$  having the interaction potential  $V_{int}$ ,

$$V_{int} = \sum_{k=2}^{\infty} \epsilon^{k-1} A_k \varphi^{2k}, \quad (50)$$

and the propagator  $\Delta$  (here  $K$  plays the role of the external current coupled to this hypothetical scalar  $\varphi$ ). Therefore, it can be evaluated order by order using the well known perturbation theory techniques, i.e. by calculating the Feynman diagrams corresponding the interaction potential (50). The only difference is that an extra  $\square^{-1}$  factor should be attached to each *external* line since one should set  $\square K = J$  at the end.

Consider a connected graph in  $\varphi$  theory with  $I$  internal lines,  $E$  external lines,  $N_k$  vertices of type  $\varphi^{2k}$  and  $L$  loops. For this diagram we have

$$2I + E = \sum_k N_k(2k), \quad L = I - N + 1, \quad (51)$$

where  $N = \sum_k N_k$  is the total number of interaction vertices. Upto a finite constant prefactor, the corresponding truncated function becomes

$$G \sim S \prod_{m=1}^N \prod_{l=1}^I \int d^n p_l \delta(m) \Delta(p_l), \quad (52)$$

where  $\delta(m)$  represents the Dirac delta function at the vertex  $m$ ,  $\Delta(p)$  is the momentum space propagator which can be calculated from (40) and (49) as

$$\Delta(p) = \frac{1}{p^{-2} + \frac{2a_2}{a_0}}, \quad (53)$$

and  $S$  is the contribution of the cutoff dependent coupling constants in (50):

$$S = \prod_k \epsilon^{(k-1)N_k} = \Lambda^{-n \sum_k (k-1)N_k}. \quad (54)$$

Note that the asymptotic behavior of  $\Delta$  is very different than the propagator of a conventional scalar field and we have

$$\Delta(p) < \frac{a_0}{2a_2}. \quad (55)$$

Using this inequality in (52) (and ignoring the finite parts) one finds

$$G < S \prod_{m=1}^N \prod_{l=1}^I \int d^n p_l \delta(m). \quad (56)$$

The delta functions impose linear relations among the internal momenta which reduce the number of independent momentum variables to the number of loops (of course one delta function survives which ensures conservation of external momentum). Moreover each loop integral in (56) diverges like  $\Lambda^n$  and thus

$$G < S \Lambda^{nL}, \quad (57)$$

which gives an upper bound for the *actual* degree of divergence of this diagram. Solving for  $I$  and  $L$  from (51), and using (54) one finally obtains

$$G < \Lambda^{n(1-E/2)}. \quad (58)$$

Eq. (58) shows that all graphs which have more than two external lines vanish and all corrections to the propagator are bounded in the continuum limit. Therefore, the theory becomes *trivial* as one removes the cutoff.

A few comments are now in order. Although corrections to the propagator are all finite, it is not clear if the sum of these terms would converge or if the final result would correspond to a physical particle. The first term in this series was actually calculated in (41). This corresponds to a one-loop diagram in  $\varphi$  theory which arises from the first term in the potential in (50) (see figure 1). The tree level contribution of the same term to the four-point function is suppressed by the cutoff dependence of the “coupling constant” in  $V_{int}$ . Note that in the second line of (41) the propagator  $P_2$  is attached to the external lines instead of  $\Delta$ . This is because, as pointed out above, the external lines should be multiplied by  $\square^{-1}$  in obtaining  $Z[J]$  and we have  $\square^{-1}\Delta = P_2$ .

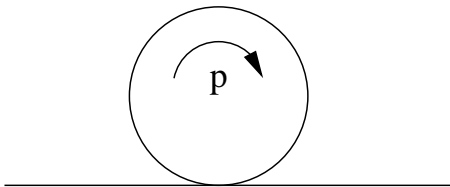


FIG. 1: The first order correction to the propagator which arises from  $\epsilon A_2 \varphi^4$  term in the potential (50). The loop integral, which is calculated with the propagator  $\Delta$  in (53), diverges like  $\Lambda^n$ . This divergence is compensated by the  $\epsilon$  factor and the result becomes finite in the  $\Lambda \rightarrow \infty$  limit.

The above argument of triviality also works for non-polynomial potentials which cannot be analyzed in usual perturbation theory. The only requirement is that the cutoff dependences of the bare parameters should be fixed as in (15) to have a well defined non-Gaussian path integral without the kinetic term. In general (15) is not valid for the bare parameters obtained in usual perturbation theory; for instance in  $\lambda\phi^4$  theory in 4-dimensions there is a one loop logarithmic  $\Lambda$  dependence in  $\lambda$ . So, the triviality argument *is not* directly applicable to perturbatively treated self-interactions.

Returning back the kinetic expansion discussed in section II, one may conclude that the infinities arise simply because locality is ignored in calculating the zeroth order contribution  $Z_0[J]$  and, oppositely, the corrections demand locality (this viewpoint was emphasized in [15]). Therefore, to be able to use (18) one may try to find a way of resolving this dilemma. One possibility is to set up an RG flow starting from IR as discussed in footnote 3. It would be interesting to verify triviality directly from (18) by using this RG flow, so that the locality can be recovered step by step starting from IR.

## V. CONCLUSIONS

In this paper, an alternative expansion scheme is obtained in self-interacting scalar field theories which uses the elements of kinetic expansion and usual perturbation theory together. The central object in this framework is the non-Gaussian functional integral of the potential. These integrals naturally arise in the kinetic expansion, which, however, is plagued by the presence of infinities. Kinetic expansion is ill defined since locality is completely neglected in the zeroth order configuration around which there is a local expansion. To resolve the problem we introduced a Lagrange multiplier to separate the kinetic and the potential terms in the path integral. In this way, we are able to invert the kinetic term while the potential can be integrated out non-perturbatively but order by or-

der in a cutoff parameter yielding a series expansion for the generating functional. Although we just consider a real scalar field, it is not difficult to generalize the formulas for coupled multiple scalars. For instance, the analog of the one dimensional integral (8) is a multi-dimensional one fixed by the number of scalars in the theory.

Using this new scheme, we show that if the bare parameters in the potential are chosen to have a well defined non-Gaussian path integral then all graphs except the two-point function vanish in the continuum limit. Therefore, the theory becomes trivial when the cutoff is removed. Note that the bare couplings are fixed “kinematically”, i.e. to define a proper integration measure. For polynomial potentials it should be possible to give a proof of triviality in usual perturbation theory as suggested by the naive power counting discussed in section IV. However, it is difficult to see triviality in the kinetic expansion.

Changing the potential only modifies the constants  $a_m$  (i.e. their dependences on the parameters of the theory) and the other steps in the calculations remain unaffected. This indicates that the self-interaction potential simply plays the role of a weight function in the path integral. Note that in Euclidean space the convergence of the generating functional (1) should be more rapid for “larger” interaction potentials since contribution of each field configuration in the functional integral would decrease due to the  $\exp(-\int V)$  factor. This can be observed in our approach since the path integral containing the potential reduces to product of ordinary integrals. However, this is contrary to what we see in usual perturbation theory; for example in four-dimensions although  $\lambda\phi^4$  theory is perturbatively tractable  $\lambda\phi^6$  is not.

It would be interesting to extend the present work in different directions. Firstly, it is possible to combine the above formalism with the usual perturbation theory. This can be achieved by taking *all* bare quantities in the Lagrangian (the wave-function, the coupling constants etc.) as provided by the perturbative renormalization theory and use them in the expression (31) or (32). This may be helpful in addressing puzzles like the hierarchy problem. Secondly, one may consider non-Gaussian path integrals for self-interacting fermions. Finally, it would be interesting to study scalars coupled to gauge fields and, especially, to analyze the mechanism of spontaneous symmetry breaking using non-Gaussian path integrals.

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